Problem 1
Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous bijection such that $f^{-1} : \mathbb{R} \to \mathbb{R}$ is also continuous.

1. Show that a sequence (a_n) converges if and only if the sequence $(f(a_n))$ converges. If $(f(a_n))$ converges to L, what must (a_n) converge to?
2. Give an example of a continuous function g and a sequence (b_n) such that $(g(b_n))$ converges but (b_n) does not converge.

Problem 2

1. Suppose $(a_n), (b_n), (c_n)$ are sequences of real numbers, such that for all $n \in \mathbb{N}$, $a_n \leq b_n \leq c_n$. If (a_n) and (c_n) converge and
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$
show that (b_n) converges and $\lim_{n\to\infty} b_n = L$.
2. Show the same is true if we replace "for all $n \in \mathbb{N}$, $a_n \leq b_n \leq c_n$" with "there exists $M \in \mathbb{N}$ such that $n \geq M$ implies $a_n \leq b_n \leq c_n$."

Problem 3

1. Prove that for all $x \in (0, \infty)$,
$$\lim_{n\to\infty} \frac{nx}{n+x} = x.$$
(*Hint*: Factor nx from the denominator.)
2. Prove that, for all $x \in (0, \infty)$,
$$\log\left[\left(1 + \frac{x}{n}\right)^n\right] = n \int_n^{n+x} \frac{1}{t}\, dt.$$
3. Prove that, for all $x \in (0, \infty)$,
$$\lim_{n\to\infty} \log\left[\left(1 + \frac{x}{n}\right)^n\right] = x.$$
(*Hint*: use problem 1 and 2.)
4. Compute $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$. (*Hint*: use problem 1.)

Problem 4
Suppose (a_n) is a bounded sequence of real numbers. Define

$$(s_n) = \sup_{k \geq n} a_k = \sup\{a_k : k \geq n\},$$
$$(t_n) = \inf_{k \geq n} a_k = \inf\{a_k : k \geq n\}.$$

1. Show that (s_n) is monotone nonincreasing and (t_n) is monotone nondecreasing. Conclude that they both converge.

   Remark: We write

   $$\lim_{n \to \infty} s_n = \limsup_{n \to \infty} a_n,$$
   $$\lim_{n \to \infty} t_n = \liminf_{n \to \infty} a_n.$$

   They are called the "limit superior" and "limit inferior" of (a_n), respectively.

2. Show that (a_n) converges to $L \in \mathbb{R}$ if and only if

   $$\limsup_{n \to \infty} a_n = \liminf_{n \to \infty} a_n = L.$$

Problem 5
Fix some number $x > 0$. Define a sequence (a_n) as follows: let $a_1 = 1$, and for all $n > 1$ define a_n recursively as

$$a_n = \frac{1}{2}\left(a_{n-1} + \frac{x}{a_{n-1}}\right).$$

1. Prove that for all $n \in \mathbb{N}$, $a_n > 0$.
2. Prove that for all $t \in (0, \infty)$,

   $$t + \frac{1}{t} \geq 2.$$

   Use this to show that for $n \geq 2$, $a_n \geq \sqrt{x}$. (*Hint*: factor out $\sqrt{x}$ from the parentheses.)
3. Prove that (a_n) is non-increasing for $n \geq 2$. Conclude that (a_n) converges. (*Hint*: prove that for $n \geq 2$, $\frac{a_{n+1}}{a_n} \leq 1$.)
4. Compute $\lim_{n \to \infty} a_n$.